DG-SG method for Maxwell Equation

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1.DG method

For the simple example of partial differential equation,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = f$$

First multiply both side by a test function v, and integrate it over the domain K,

$$\int_K \frac{\partial u}{\partial t} v + \int_K \frac{\partial u}{\partial x} v = \int_K fv$$

Then we can use the integration by parts formula,

$$\int_K \frac{\partial u}{\partial t} v + \int_{\partial K} \hat{u} v - \int_K u \frac{\partial v}{\partial x} = \int_K fv$$

Since we are using the orthonormal basis function ϕ, we can represent u, v, f by

$$u = \sum_i u_i \phi_i \quad v = \sum_i \phi_i \quad f = \sum_i f_i \phi_i$$

And we can denote $[u_1, u_2, ...]^{\mathrm{T}}$ by u_h, and $[f_1, f_2, ...]^{\mathrm{T}}$ by f_h

Since the basis function set is orthonormal, if $i \neq j$,

$$\int_K \phi_i \phi_j = 0$$

Then we can change

$$\int_K \frac{\partial u}{\partial t} v \quad to \quad \frac{d}{dt} I u_h, \qquad \int_K f v \quad to \quad I f_h$$

where I is the identity matrix

Here we use the central flux

$$\hat{u} = \frac{1}{2}(u^+ + u^-)$$

And we can also convert

$$\int_{\partial K} \frac{1}{2}(\phi_i^+ + \phi_i^-)\phi_j - \int_K \phi_i \frac{\partial \phi_j}{\partial x}$$

to matrix A using Gaussian quadrature formula

Then the partial differential equation can be changed to the linear multiplication,

$$\frac{d}{dt} u_h = -A * u_h + f_h$$

Hence we can use the explicit time advanced method

$$\frac{u_h^{(t+1)} - u_h^{(t)}}{\Delta t} = -A * u_h^{(t)} + f_h$$

2.Sparse Grid

We consider the interval $\Omega = [0,1]$, then the n-th level grid Ω_n consists of 2^n uniform cells

$$I_j^n = (2^{-n}j, 2^{-n}(j+1)), j = 0, ..., 2^n - 1$$

we can define

$$V_k^n := \{v : v \in P^k(I_j^n), \forall j = 0, ..., 2^n - 1\}$$

to be the usual piecewise polynomials of degree at most k. Then we have

$$V_k^0 \subset V_k^1 \subset V_k^2 \subset ...$$

We can now define the multiwavelet subspace W_k^n, $n = 1,2,...$

$$V_k^{n-1} \bigoplus W_k^n = V_k^n, W_k^n \perp V_k^{n-1}$$

as the orthogonal complement of V_k^{n-1} in V_k^n with respect to the L^2 inner product

Now, for a multi-index $l = [l_1, l_2, ..., l_d] \in \mathbb{N}_0^d$, the l^1 and l^∞norms are defined as

$$|\boldsymbol{\alpha}|_1 := \sum_{m=1}^{d} \alpha_m, |\boldsymbol{\alpha}|_\infty := \max_{1 \le m \le d} \alpha_m$$

where d is the dimension and $\mathbb{N}_0^d$ denotes the set of nonnegative integers

We define the tensor-product mesh grid

$$\Omega_l = \Omega_{l_1} \bigotimes \Omega l_2 \bigotimes ... \bigotimes \Omega_{l_d}$$

Based on the tensor product construction, we have

$$\boldsymbol{W}_k^l = W_{k,x_1}^{l_1} \times W_{k,x_2}^{l_2} \times ... \times W_{k,x_d}^{l_d}$$

Also based on the one-dimensional hierarchical decomposition, we have

$$\boldsymbol{V}_k^l = V_{k,x_1}^{l_1} \times V_{k,x_2}^{l_2} \times ... \times V_{k,x_d}^{l_d} = \bigoplus_{j_1 \le l_1, ..., j_d \le l_d} \boldsymbol{W}_k^j$$

$$\boldsymbol{V}_k^N = V_{k,x_1}^N \times V_{k,x_2}^N \times ... \times V_{k,x_d}^N = \bigoplus_{|l|_\infty \le N} \boldsymbol{W}_k^j$$

Where $\boldsymbol{V}_k^N$ is denoted if we use $h_N = 2^{-N}$ in each direction

And the sparse finite element approximation space $\hat{\boldsymbol{V}}_k^N$ is defined on Ω_N by

$$\hat{\boldsymbol{V}}_k^N = \bigoplus_{|l|_1 \leq N} \boldsymbol{W}_k^j$$

The degrees of freedom of sparse grid space scales as

$$\mathcal{O}((k+1)^d 2^N N^{d-1})$$

which is significantly less than that of traditional space with

$$\mathcal{O}((2^N(k+1))^d)$$

This is the key for computational savings and reduction in high dimensions.

3.Maxwell Equation

As for the Maxwell Equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0$$

$$\epsilon\mu\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} = -\mu\boldsymbol{J}$$

And for piecewise functions, we introduce the jumps and averages as follows

For any edge $e = K^+ \cap K^-$, with $\boldsymbol{n}^\pm$ as the outward unit normal to $\partial K^\pm$

the jumps across e are defined as

$$[\boldsymbol{U}]_{\boldsymbol{n}} = \boldsymbol{U}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{U}^- \cdot \boldsymbol{n}^-$$

$$[\boldsymbol{U}]_\tau = \boldsymbol{U}^+ \times \boldsymbol{n}^+ + \boldsymbol{U}^- \times \boldsymbol{n}^-$$

and the averages are

$$\{\boldsymbol{U}\} = \frac{1}{2}(\boldsymbol{U}^+ + \boldsymbol{U}^-)$$

Use the similar idea we introduced above

the semi-discrete DG methods for Maxwell equation are defined by following

$$\int_K \partial_t \boldsymbol{B}_h \cdot \boldsymbol{V}\ d\boldsymbol{x} = -\int_K \boldsymbol{E}_h \cdot \nabla \times \boldsymbol{V}\ d\boldsymbol{x} - \int_{\partial K} \widehat{\boldsymbol{n} \times \boldsymbol{E}_h} \cdot \boldsymbol{V}\ ds$$

$$\int_K \epsilon\mu\partial_t \boldsymbol{E}_h \cdot \boldsymbol{V}\ d\boldsymbol{x} = \int_K \boldsymbol{B}_h \cdot \nabla \times \boldsymbol{U}\ d\boldsymbol{x} + \int_{\partial K} \widehat{\boldsymbol{n} \times \boldsymbol{B}_h} \cdot \boldsymbol{U}\ ds - \int_K \mu \boldsymbol{J} \cdot \boldsymbol{U}\ d\boldsymbol{x}$$

where $\boldsymbol{E}_h, \boldsymbol{B}_h, \boldsymbol{U}, \boldsymbol{V}$ are from the wavelet sparse grid basis

and all hat functions denote the numerical fluxes.

4. Numerical Flux

$$central\ flux: \quad \hat{\boldsymbol{E}}_h = \frac{1}{2}(\boldsymbol{E}_h{}^+ + \boldsymbol{E}_h{}^-),\ \hat{\boldsymbol{B}}_h = \frac{1}{2}(\boldsymbol{B}_h{}^+ + \boldsymbol{B}_h{}^-)$$

$$alternating\ flux: \quad \hat{\boldsymbol{E}}_h = \boldsymbol{E}_h{}^+,\ \hat{\boldsymbol{B}}_h = \boldsymbol{B}_h{}^-\ or\ \hat{\boldsymbol{E}}_h = \boldsymbol{E}_h{}^-,\ \hat{\boldsymbol{B}}_h = \boldsymbol{B}_h{}^+$$

$$upwind\ flux: \quad \hat{\boldsymbol{E}}_h = \{\boldsymbol{E}_h\} + \frac{1}{2}[\boldsymbol{B}_h]_\tau,\ \hat{\boldsymbol{B}}_h = \{\boldsymbol{B}_h\} - \frac{1}{2}[\boldsymbol{E}_h]_\tau$$

Let $\boldsymbol{U} = \boldsymbol{E}_h$, $\boldsymbol{V} = \boldsymbol{B}_h$,

$$-\sum_K \int_K \mu \boldsymbol{J} \cdot \boldsymbol{E}_h \, d\boldsymbol{x} = \frac{1}{2}\frac{d}{dt}\sum_K \Big(\int_K (\epsilon\mu|\boldsymbol{E}_h|^2 + |\boldsymbol{B}_h|^2) \, d\boldsymbol{x} - \int_{\partial K} ([\boldsymbol{E}_h \times \boldsymbol{B}_h] + \hat{\boldsymbol{B}}_h \times [\boldsymbol{E}_h]_\tau - \hat{\boldsymbol{E}}_h \times [\boldsymbol{B}_h]_\tau) \, ds \Big)$$

And since for central flux and alternating flux,

$$[\boldsymbol{E}_h \times \boldsymbol{B}_h] + \hat{\boldsymbol{B}}_h \times [\boldsymbol{E}_h]_\tau - \hat{\boldsymbol{E}}_h \times [\boldsymbol{B}_h]_\tau = 0$$

Then we have energy conservation

$$-\sum_K \int_K \mu \boldsymbol{J} \cdot \boldsymbol{E}_h \, d\boldsymbol{x} = \frac{1}{2}\frac{d}{dt}\sum_K \Big(\int_K (\epsilon\mu|\boldsymbol{E}_h|^2 + |\boldsymbol{B}_h|^2) \, d\boldsymbol{x}$$

And for upwind flux, the energy is decreasing,

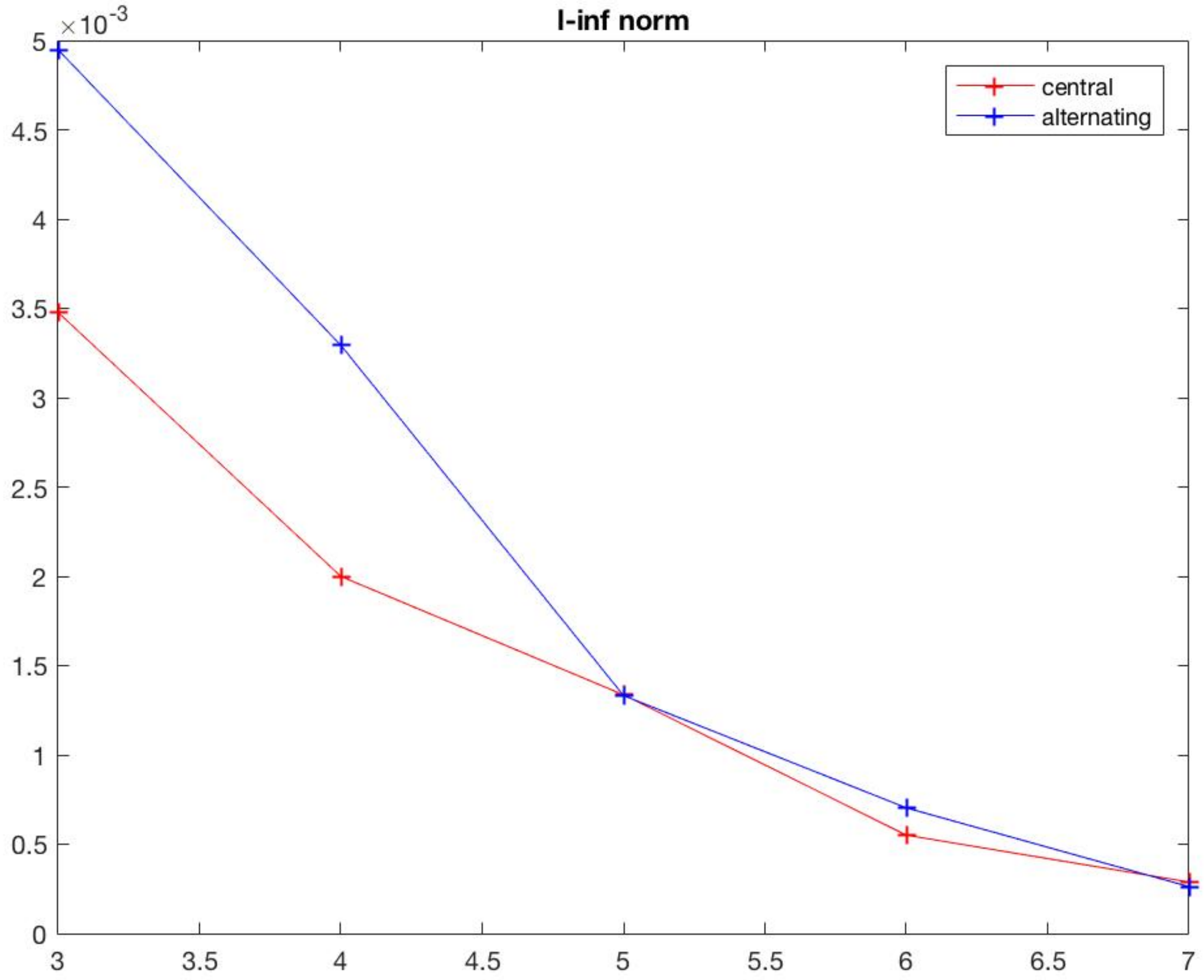
$$-\sum_K \int_K \mu \boldsymbol{J} \cdot \boldsymbol{E}_h \, d\boldsymbol{x} = \frac{1}{2}\frac{d}{dt}\sum_K \Big(\int_K (\epsilon\mu|\boldsymbol{E}_h|^2 + |\boldsymbol{B}_h|^2) \, d\boldsymbol{x} + \frac{1}{2}\int_{\partial K} |[\boldsymbol{E}_h]_\tau|^2 + |[\boldsymbol{B}_h]_\tau|^2 \, ds \Big)$$

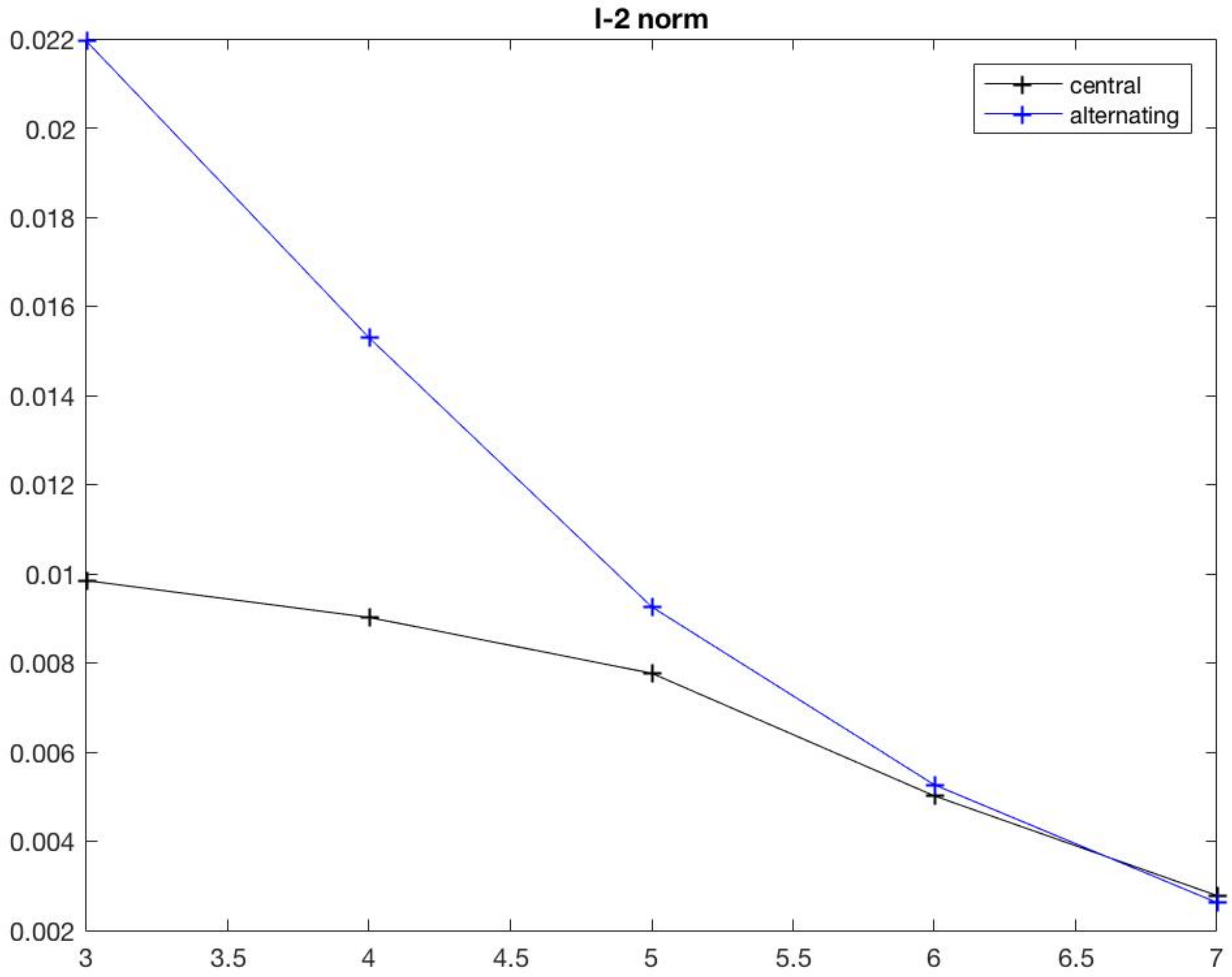
Table 1: The accuracy of order for central flux

deg	level	$l-\infty$ error	order	$l-2$ error	order
2	3	3.477E-03		9.847E-03	
2	4	1.999E-03	0.799	9.020E-03	0.127
2	5	1.336E-03	0.581	7.767E-03	0.216
2	6	5.509E-04	1.279	5.017E-03	0.631
2	7	2.882E-04	0.935	2.781E-03	0.851

Table 2: The accuracy of order for alternating flux

deg	level	$l-\infty$ error	order	$l-2$ error	order
2	3	4.948E-03		2.195E-02	
2	4	3.293E-03	0.587	1.529E-02	0.521
2	5	1.332E-03	1.306	9.257E-03	0.724
2	6	7.036E-04	0.921	5.265E-03	0.814
2	7	2.626E-04	1.422	2.623E-03	1.006


l-inf norm
×10^-3
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
3
3.5
4
4.5
5
5.5
6
6.5
7
central
alternating



l-2 norm
central
alternating
0.022
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
3
3.5
4
4.5
5
5.5
6
6.5
7